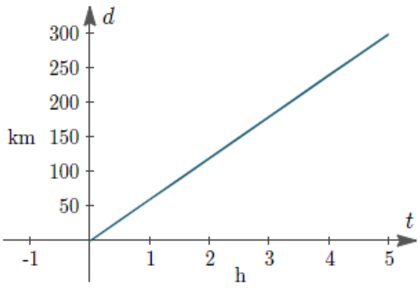
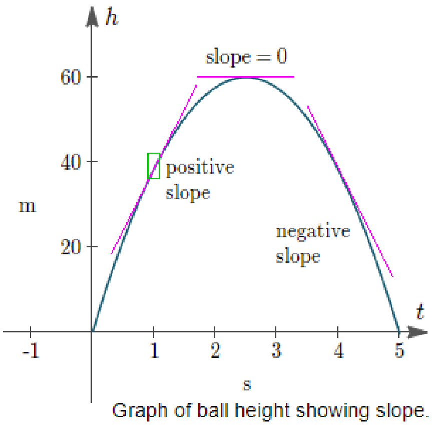
Differentiation

Differentiation in calculus deals with finding the rates of change of any quantity in relation to another when the rate is not constant.



Graph of $d = 60t$.

Figure 1 is what we see when we graph the distance travelled by a car from the starting point versus the time taken to cover it. The slope of the line is always positive and there is a constant rate of change.



Graph of ball height showing slope.

But **Figure 2** is what we get when we throw a ball straight up in the air and plot the height of the ball versus the time taken for it to cover the distance. Gravity slows the ball down as it's going up and it reverses direction and begins to fall faster. So the slope is changing throughout the ball's journey. It has a positive slope when we throw it, decreases as the ball slows and becomes 0 at the ball's highest point and then decreases further steeply until the ball hits the ground.

Differentiation has many applications in science and engineering. It is especially useful in optimization where conditions for a maximum or minimum are sought. For instance, in business, to find the conditions for cost reduction, profit increase, and in engineering, the conditions for maximum strength, minimum cost etc.

There are some **rules of differentiation** for two functions **u** and **v**:

1) **Addition and subtraction**: $(u \pm v)' = u' + v'$
   e.g., $d(5 + 12x - x^3)/dx = d(5)/dx + d(12x)/dx - d(x^3)/dx = 0 + 12 - 3x^2 = 12 - 3x^2$
2) **Product rule**: $(uv)' = u'v + uv'$
   e.g., $d((3x-1)(2x+3))/dx = (2x+3)d(3x-1)/dx + (3x-1)d(2x+3)/dx = (2x+3)[d(3x)/dx - d(1)/dx] + (3x-1)[d(2x)/dx+d(3)/dx] = (2x+3)(3-0)+(3x-1)(2) = 6x+9+6x-2 = 12x+7$
3) **Quotient rule**: $(u/v)' = (u'v - uv')/v^2$

4) **Chain rule**: If **f** is a composite of **u** and **v (f = v o u)**, **t = u(x)**, and **dt/dx** and **dv/dt** both exist,
$df/dx = (dv/dt)*(dt/dx)$ OR if $F(x) = f(g(x))$, then $F'(x) = f'(g(x))* g'(x)$
Let's look at an example that combines the quotient and chain rules.
$d(\sin(2x)/(2x))/dx = ?$
$\sin(2x)$ is a composite function. $d(\sin(2x)/dx = \cos(2x)*2$

$$d(\sin(2x)/(2x))/dx = (\frac{(\cos(2x)*2)(2x) - \sin(2x)*(2)}{(2x)^2} = \frac{2x\cos(2x) - \sin(2x)}{2x^2}$$

Sometimes **y** cannot be expressed explicitly in terms of **x** alone. In such cases, y cannot be solved for y by finding dy/dx. To find the rate of change of y as x changes, we need to turn to **implicit differentiation**.
e.g., Find dy/dx if $y^4 + 2x^2 y^2 + 6x^2 = 7$
This combines many rules (addition, product).
Let us take derivative of both sides.
$d(y^4)/dx + d(2 x^2 y^2)/dx + d(6x^2)/dx = d(7)/dx$
$4y^3 dy/dx + (2x^2)d(y^2)/dx + (y^2)d(2x^2)/dx + 12x = 0$
$4 y^3 dy/dx + (2x^2)(2y)dy/dx + (y^2)(4x) + 12x = 0$
$4 y^3 dy/dx + 4x^2y dy/dx + 4 xy^2 + 12x = 0$
$(4y^3 + 4x^2y) dy/dx = -4xy^2 - 12x$
$dy/dx = (-4xy^2 -12x)/(4y^3 + 4x^2y) = (-xy^2 - 3x)/(y^3 + x^2y)$

Higher derivatives: If you take the derivative of the first derivative, you get the second derivative. And the derivative of the second derivative gives the third derivative, and so on.
e.g., Find the higher derivatives of $y = x^5 + 3x^3 - 2 x + 7$
First derivative: $dy/dx = y' = 5x^4 + 9 x^2 - 2$
Second derivative: $d^2y/dx^2 = y'' = 20x^3 + 18x$
Third derivative: $d^3y/dx^3 = y''' = 60x^2 + 18$
Fourth derivative: $d^4y/dx^4 = y^{iv} = 120x$
Fifth derivative: $d^5y/dx^5 = y^v = 120$
All higher derivatives (6^{th} onwards) are 0.

Partial derivatives: Finding a partial derivative with respect to x means differentiate the x parts and consider all other letters as constants.
e.g., Suppose $F(x,y) = y + 6 \sin x + 5 y^2$, where F is a function of two variables, x and y. Find the partial derivative of F with respect to x.
We plot F(x,y) and get the graph seen in **Figure 3**. Finding the partial derivative with respect to x means we should look at the x-z plane of the graph from the far end of the y axis (**Figure 4**). There is a 6 sin x component in the equation and we see a sine curve along the x-axis. Considering the y components as constants, the partial derivative of $\partial F/\partial x = 6 \cos x$

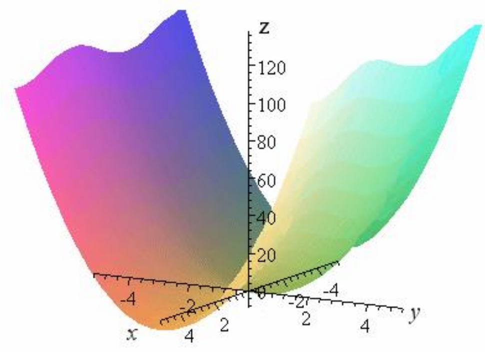


Figure 3

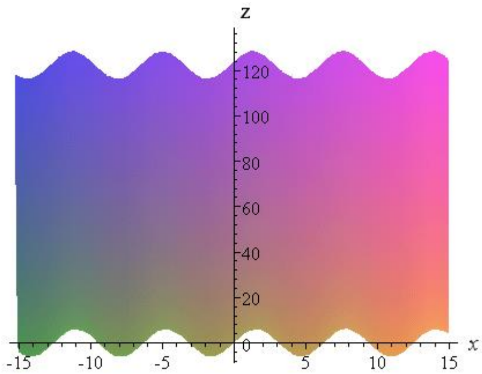


Figure 4

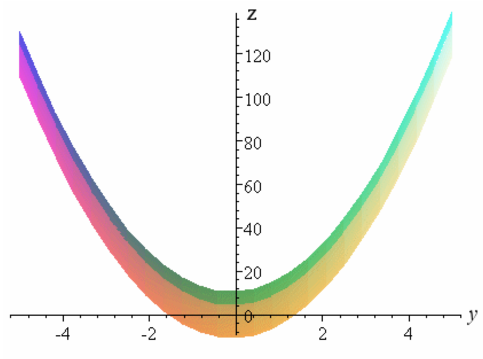


Figure 5

To find the partial derivative of F(x,y) with respect to y, we differentiate the y parts but consider all other letters as constants. Now, we turn the graph and look at it from the far end of the x-axis at the y-z plane (**Figure 5**). Remember that if we consider the x component to be constant, F is $y + 5y^2$ which is a parabola.

$\partial F/\partial y = 1 + 10\ y$

We can find the second-order partial derivative, $\partial^2 F/\partial y\partial x$ of the above function F(x,y).

$\partial^2 F/\partial y\partial x = \partial\ (\partial F/\partial x)/\ \partial y$ (first, we find the partial derivative of F with respect to x, and then find the partial derivative of the result with respect to y)

$\partial F/\partial x = 6 \cos x$ so $\partial^2 F/\partial y\partial x = \partial\ (\partial F/\partial x)/\ \partial y = \partial\ (6 \cos x)/\ \partial y = 0$

Remember when we consider differentiation with respect to y, cos x is a constant.

$\partial 2F/\partial x\partial y = \partial\ (\partial F/\partial y)/\ \partial x = \partial\ (1 + 10\ y)/\ \partial x = 0$

$\partial 2F/\partial x^2 = \partial\ (\partial F/\partial x)/\ \partial x = \partial\ (6 \cos x)/\ \partial x = -6 \sin x$

$\partial 2F/\partial y^2 = \partial\ (\partial F/\partial y)/\ \partial y = \partial\ (1+10y)/\ \partial y = 10$